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Analytic calculation of color-Coulomb potential

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We develop a calculational scheme in Coulomb and temporal gauge that respects gauge invariance and is most easily applied to the infrared asymptotic region of QCD. It resembles the Dyson-Schwinger equations of Euclidean quantum field theory in Landau gauge, but is 3-dimensional. A simple calculation yields a color-Coulomb potential that behaves at large R approximately like $V_{\mathrm{coul}}(R) \sim R^{[1-0.2(d-1)]}$ for spatial dimension $1 \leq d \leq 3$. This is a linearly rising potential plus a rather weak dependence on d.

1. Introduction

There is a simple confinement scenario in Coulomb gauge [1], [2] which, in short, attributes confinement to the long range of the color-Coulomb potential, $V_{\rm coul}(R)$. This quantity is the instantaneous part of the 00-component of the dressed gluon propagator in minimal Coulomb gauge,[1]

$$g_0^2 D_{00}(\vec{x}, x_0) = \langle g A_0^a(\vec{x}, x_0) g A_0^b(0,0) \rangle = V_{\rm coul}(|\vec{x}|)\delta(x_0) + ({\rm non-instantaneous}), \quad (1.1)$$

and is given by [3]

$$V_{\rm coul}(x-y)\delta^{ab} = \langle\ g_0^2[M^{-1}(A)(-\partial^2)M^{-1}(A)]_{xy}^{ab}\ \rangle. \quad (1.2)$$

Here $M(A) \equiv -\partial_i D_i(A)$ is the Faddeev-Popov operator, and the gauge-covariant derivative is defined by $[D_i(A)\omega]^a = \partial_i \omega^a + g_0 f^{abc} A_i^b \omega^c$.

We present a calculation of $V_{\rm coul}(R)$. This quantity is of interest because: (i) It couples universally to color charge. (ii) Confinement of color-charge may be explained by the long range of this potential. (iii) It is a renormalization-group invariant and is independent of the cut-off and the renormalization mass [2]. (iv) A necessary condition for the Wilson potential $V(R)$ to be confining is that $V_{\rm coul}(R)$ be confining [4], and if both potentials rise linearly at large R, $V(R) \sim \sigma R$ and $V_{\rm coul}(R) \sim \sigma_{\rm coul} R$, then $\sigma_{\rm coul} \geq \sigma$. (v) We wish to compare with a recent numerical determination [5] of $V_{\rm coul}(R)$, that does show a linear rise at large R, with $\sigma_{\rm coul} \sim 3\sigma$.

Calculations in the Coulomb gauge have been pursued vigorously. For recent work and further references, see [6]. The present approach is distinguished by particular attention to gauge invariance, and its easiest application is to the infrared asymptotic limit of QCD.

2. Temporal gauge and Coulomb gauge

For simplicity we consider pure gluodynamics. In the temporal or Weyl gauge, $A_0 = 0$, the wave functionals $\Psi(A)$ depend on $A_i^a(x)$ for $i = 1, 2, 3$. The color-electric field operator is represented by $E_i^a(x) = i\delta/\delta A_i^a(x)$, and the hamiltonian by

$$H \equiv \frac{1}{2} \int d^3x\ (E^2 + B^2), \quad (2.1)$$

[1] In this equation $\vec{x}$ represents a 3-vector, but everywhere else in this article 3-vectors are represented by x.

where $B_i^a = \epsilon_{ijk}(\partial_j A_k^a + \frac{1}{2}g_0 f^{abc} A_j^b A_k^c)$, and the f^{abc} are the structure constants of the Lie algebra of the SU(N) group. Wave functionals in temporal gauge are required to be gauge invariant $\Psi({}^g A) = \Psi(A)$, where $g(x) \in SU(N)$ is a 3-dimensional local gauge transformation, and ${}^g A_i \equiv g_0^{-1} g^{-1} \partial_i g + g^{-1} A_i g$. These continuum equations have precise analogs in lattice gauge theory, where the Kogut-Suskind hamiltonian replaces the Weyl hamiltonian.

Poincare invariance of the continuum theory is preserved because the hamiltonian density $T^{00} = \frac{1}{2}(E^2 + B^2)$ satisfies the Dirac-Schwinger equal-time commutation relation

$$[T^{00}(x), T^{00}(y)] = -i[T^{0i}(x) + T^{0i}(y)]\partial_i \delta(x-y) + \text{S.T.}, \tag{2.2}$$

where $T^{0i} = \frac{1}{2}\epsilon_{ijk}(E_j^a B_k^a + B_k^a E_j^a)$ is the Poynting vector [7], and S.T. is the Schwinger term.

Inner products in temporal gauge, $(\Psi_1, \Psi_2) = N \int dA \ \Psi_1^*(A)\Psi_2(A)$, are divergent because of the gauge invariance of the wave-functionals. They may be made finite by using the Faddeev-Popov identity

$$(\Psi_1, \Psi_2) = \int_\Lambda dA^{\text{tr}} \ \det M(A^{\text{tr}}) \ \Psi_1^*(A^{\text{tr}})\Psi_2(A^{\text{tr}}). \tag{2.3}$$

The integral extends over 3-dimensionally transverse configurations in the fundamental modular region Λ, which is a region free of Gribov copies. To be definite we suppose that we are in the minimal Coulomb gauge, which is obtained by minimizing $F_A(g) = ||{}^g A||^2$ with respect to gauge transformations $g(x)$, so $||A|| \leq ||{}^g A||$ for all $g(x)$ and all A in Λ.

Wave-functionals in minimal Coulomb gauge $\Psi(A^{\text{tr}})$ are the restriction of gauge-invariant wave functionals in temporal gauge $\Psi(A)$ to the fundamental modular region Λ. Conversely every wave-functional in minimal Coulomb gauge has a unique gauge-invariant extension to temporal gauge. Every point A in the interior of Λ, is a unique absolute minimum (modulo global gauge transformations), so the strict inequality holds $||A|| < ||{}^g A||$ (for all $g(x)$ that is not a global gauge transformation). But every point A_1 on the boundary $\partial\Lambda$ of Λ is related by a local gauge transformation $g(x)$ to some other point $A_2 = {}^g A_1$ also on $\partial\Lambda$, with which it is degenerate, $||A_1|| = ||A_2||$. This gauge transformation may be infinitesimal, $A_2 = A_1 + \epsilon D(A_1)\omega$, where $D(A_1)\omega$ is tangent to $\partial\Lambda$. Gauge-invariance requires that the wave-functional in Coulomb gauge be identified at corresponding boundary points, $\Psi(A_2) = \Psi(A_1)$, or, for the infinitesimal case, that the wave functional satisfies $(D(A_1)\omega, \frac{\delta\Psi}{\delta A}|_{A_1}) = 0$. This provides the boundary condition that is needed to make the

hamiltonian in Coulomb gauge well-defined and symmetric. The identification of boundary points is often ignored because one does not know explicitly what the boundary of Λ is. But in general it would be a violation of gauge invariance to ignore this identification and take arbitrary wave-functionals in so-called physical coordinates which are the transverse configurations in Λ. In order not to make this error, in the present article we shall use wave functionals $\Psi(A)$ that are manifestly gauge-invariant.

As an example we exhibit an approximate vacuum wave functional that is gauge invariant. The variation of the color-magnetic field is given by

$$\delta B_i^a = \epsilon_{ijk} D_j^{ac} \delta A_k^c \equiv (\hat{D}\delta A)_i^a, \tag{2.4}$$

which defines the hermitian operator $\hat{D}(A)$ that is the gauge-covariant curl. Consider the wave functional

$$\Phi = \exp\Big(-\frac{1}{2} \int d^3x \; B_i^a \; [(\hat{D}^2)^{-1/2} B]_i^a \Big). \tag{2.5}$$

The operator $\hat{D}(A)$ has small eigenvalues when acting on longitudinal fields, but the Bianchi identity $D_i B_i = 0$ insures that the wave-functional is regular. We have $\frac{\delta \Phi}{\delta A_i} \approx -[\hat{D}(\hat{D}^2)^{-1/2} B]_i \Phi$, and

$$-\frac{1}{2} \int d^3x \; \frac{\delta^2 \Phi}{\delta A_i^2} \approx \int d^3x \; (-\frac{1}{2} B^2 + f) \; \Phi, \tag{2.6}$$

where $f(x) \equiv \frac{1}{2}[(\hat{D}^2)^{1/2}]_{ii}^{aa}(x,y)|_{y=x}$, and $\approx$ means that derivatives with respect to $(\hat{D}^2)^{-1/2}$ are neglected. The first term will cancel the magnetic energy density, which is the most singular term, being the product of quantum fields at the same point. We have $f(x) = e + u(x,A)$, where $e = \frac{1}{2}[(\hat{\partial}^2)^{1/2}]_{ii}^{aa}(x,y)|_{y=x}$ is a divergent constant, and $u(x,A)$ is a gauge-invariant non-local functional that vanishes with A. The Schrödinger equation reads

$$H\Phi \approx \int d^3x \; [e + u(x,A)]\Phi, \tag{2.7}$$

and is violated by non-local terms only.

3. Calculational scheme

The vacuum wave functional $\Psi_0(A)$ is positive, and we write $\Psi_0(A) = \exp[-S(A)/2]$, where $S(A)$ is manifestly gauge invariant. We assume that $S(A)$ is either an approximate

expression, such as the one given above, or a trial expression. With gauge-invariant wave-functionals it is difficult to evaluate matrix elements by direct integration, and we shall borrow techniques from Euclidean quantum field theory. We have $|\Psi_0(A)|^2 = \exp[-S(A)]$, and we define the generating functional of equal-time correlators,

$$Z(J) \equiv \int_\Lambda dA^{\rm tr} \exp(J, A^{\rm tr}) \ \det M(A^{\rm tr}) \ \exp[-S(A^{\rm tr})], \tag{3.1}$$

normalized to $Z(0) = 1$. This is precisely the formula for the partition function or generating functional of 3-dimensional Euclidean gauge theory in the minimal Landau gauge, with the Yang-Mills Euclidean action $S_{\rm YM}(A)$ replaced by some gauge-invariant action $S(A)$. Only the transverse part of the source $J_i^a(x)$ contributes, and we take J_i to be identically transverse, $\partial_i J_i = 0$, and $J_i = J_i^{\rm tr}$.

We don't have an explicit expression for Λ, and we rely on the argument of [8] that fundamental modular region Λ and the Gribov region Ω have the same moments or correlators so we may integrate over Ω instead of Λ,

$$Z(J) \equiv \int_\Omega dA^{\rm tr} \exp(J, A^{\rm tr}) \ \det M(A^{\rm tr}) \ \exp[-S(A^{\rm tr})]. \tag{3.2}$$

Whereas Λ is the set of absolute minima of the minimizing functional, the Gribov region Ω is the set of relative minima. The matrix of second derivatives of the minimizing functional is the Faddeev-Popov operator $M(A)$. It is a non-negative matrix at a relative minimum, so the Gribov region Ω is the set of transverse configurations $A^{\rm tr}$ for which all eigenvalues $\lambda_n(A^{\rm tr})$ of $M(A^{\rm tr})$ are non-negative, $\lambda_n(A^{\rm tr}) \geq 0$. The interior of Ω consists of points $A^{\rm tr}$ where all eigenvalues are strictly positive, $\lambda_n(A^{\rm tr}) > 0$ (apart from a trivial null eigenvector corresponding to global gauge transformations). Its boundary $\partial\Omega$ consists of points where $M(A^{\rm tr})$ has a non-trivial null-eigenvector, $M(A^{\rm tr})\omega = 0$, so $\lambda_1(A^{\rm tr}) = 0$, and all other eigenvalues are non-negative, $\lambda_n(A^{\rm tr}) \geq 0$, for $A^{\rm tr} \in \partial\Omega$.

We don't have an explicit expression for Ω either, but we may exploit the fact that the integrand of (3.2) vanishes on $\partial\Omega$ to derive the Dyson-Schwinger (DS) equations nevertheless [9]. Indeed the Faddeev-Popov determinent vanishes for $A^{\rm tr} \in \partial\Omega$, $\det M(A^{\rm tr}) = \prod_n \lambda_n(A^{\rm tr}) = 0$. Thus the identity

$$0 = \int_\Omega dA^{\rm tr} \frac{\delta}{\delta A_i^{\rm tr}(x)} \Big(\exp(J, A^{\rm tr}) \ \det M(A^{\rm tr}) \ \exp[-S(A^{\rm tr})] \Big), \tag{3.3}$$

holds without a contribution from the boundary $\partial\Omega$. The set of DS equations in functional form,

$$\frac{\delta \Sigma}{\delta A_i^{\mathrm{tr},a}(x)}\Big(\frac{\delta}{\delta J}\Big)\ Z(J) = J_i^a(x)\ Z(J), \tag{3.4}$$

follow from this identity. Here $\Sigma(A^{\mathrm{tr}}) \equiv S(A^{\mathrm{tr}}) - \mathrm{tr}\ln M(A^{\mathrm{tr}})$ is the effective action. Because the integrand vanishes on $\partial\Omega$, the DS equations have the same form as they would if the integral (3.3) were extended to infinity. It is not necessary to know the boundary $\partial\Omega$ explicitly, and the cut-off at $\partial\Omega$ is implemented by imposing on the solution of the DS equations the natural positivity conditions that must be satisfied by the (equal-time) correlator $\langle A_i(x)A_j(y)\rangle$, by the ghost propagator $G(x-y) = \langle g_0 M^{-1}(A^{\mathrm{tr}})(x,y)\rangle$, and by the higher-order correlators.

As in Euclidean quantum field theory, we rewrite (3.4) as a functional DS equation for $W(J) \equiv \ln Z(J)$, which is the analog of the free energy

$$\frac{\delta \Sigma}{\delta A_i^{\mathrm{tr},a}(x)}\Big(\frac{\delta W}{\delta J} + \frac{\delta}{\delta J}\Big)\ 1 = J_i^a(x). \tag{3.5}$$

By Legendre transformation we convert this to a functional DS equation for the analog of the quantum effective action $\Gamma(A^{\mathrm{tr}}) \equiv (A^{\mathrm{tr}}, J) - W(J)$, where $A_i^{\mathrm{tr},a}(x) \equiv \frac{\delta W(J)}{\delta J_i^a(x)}$,

$$\frac{\delta \Sigma}{\delta A_i^{\mathrm{tr},a}(x)}\Big(A^{\mathrm{tr}} + \mathcal{D}\frac{\delta}{\delta A^{\mathrm{tr}}}\Big)\ 1 = \frac{\delta \Gamma}{\delta A_i^{\mathrm{tr},a}(x)}, \tag{3.6}$$

where $\mathcal{D}$ is the gluon propagator in the presence of the source, $\mathcal{D}^{-1}(A^{\mathrm{tr}}) = \frac{\delta^2\Gamma}{\delta A^{\mathrm{tr}}\delta A^{\mathrm{tr}}}$. The problem of evaluating correlators by direct functional integration has been replaced by the problem of solving the DS equations and, given the action $S(A)$, we can, at least in principle, calculate all correlators by solving the DS equations for $\Gamma(A^{\mathrm{tr}})$.

Suppose we take a trial expression $S(A,\xi)$ for the gauge-invariant action that depends on some unknown parameters ξ. These parameters are determined by minimizing $E(\xi) \equiv \langle H\rangle = (\Psi_0, H\Psi_0)$. To calculate $E(\xi)$, write $H = H_{\mathrm{e}} + H_{\mathrm{m}}$. We have $H_{\mathrm{m}} = \frac{1}{2}\int d^3x\ B^2(x)$, where $B_i^a(x) = B_i^a(x; A^{\mathrm{tr}})$, and the magnetic energy is given by

$$E_{\mathrm{m}}(\xi) = \langle H_{\mathrm{m}}\rangle = \frac{1}{2}\int d^3x\ B^2\Big(x; \frac{\delta}{\delta J}\Big)\ Z(J)|_{J=0}. \tag{3.7}$$

To calculate the electric energy

$$E_{\mathrm{e}}(\xi) = \langle H_{\mathrm{e}}\rangle = \frac{1}{2}\int d^3x \int_\Omega dA^{\mathrm{tr}}\ \det M(A^{\mathrm{tr}})\ |E(x)\Psi_0|^2, \tag{3.8}$$

we evaluate $E_i\Psi$ in temporal gauge, and then restrict to Ω. With $\Psi_0(A) = \exp[-\frac{1}{2}S(A)]$, we have $E_i^a(x)\Psi_0 = i\frac{\delta\Psi_0}{\delta A_i^a(x)} = -i\mathcal{E}_i^a(x;A)\Psi_0$, where $\mathcal{E}_i^a(x;A) \equiv \frac{1}{2}\frac{\delta S(A)}{\delta A_i^a(x)}$, which gives

$$\begin{aligned} E_{\rm e}(\xi) &= \frac{1}{2}\int d^3x \int_\Lambda dA^{\rm tr} \;\det M(A^{\rm tr})\; \mathcal{E}^2(x;A^{\rm tr}) \exp[-S(A^{\rm tr})] \\ &= \frac{1}{2}\int d^3x \; \mathcal{E}^2\Big(x; \frac{\delta}{\delta J}\Big)\; Z(J)|_{J=0}. \end{aligned} \tag{3.9}$$

Now $E(\xi) = E_{\rm e}(\xi) + E_{\rm m}(\xi)$ has, in principle, been expressed in terms of the ξ parameters that appear in $S(A,\xi)$.

4. Infrared Ansatz

This program may be difficult to carry out, especially if the action $S(A)$ is non-local. However if our experience with DS equations with action $S_{\rm YM}(A)$ is a reliable guide, then a remarkable simplification occurs in the infrared limit, as we now explain.

The DS equations with Yang-Mills action were first solved, with due attention to the ghost contribution, in [10]. The subject is reviewed in [11], and recent results are reported in [12]. We will follow the method of [13], [9]. It was found in these investigations that the ghost contribution is the dominant one in the infrared. For example, in the DS equation for the gluon propagator, the leading contribution in the infrared is provided by the gluon loop. It was subsequently realized [8] that in the DS equation (3.6), with effective action $\Sigma = S_{\rm YM} - {\rm tr}\ln M$, one obtains the correct infrared asymptotic limit by setting $S_{\rm YM} = 0$. Thus the infrared asymptotic limit is entirely determined by the Faddeev-Popov determinent $\det M(A^{\rm tr})$ and the cut-off at the Gribov horizon $\partial\Omega$. One might think that the functional integral with $S_{\rm YM} = 0$ would diverge. However the DS equations are merely a technique for evaluating the functional integral, and since they give a finite result with $S_{\rm YM} = 0$, it appears that cut-off at the Gribov horizon makes the functional integral converge.

Infrared Ansatz: We shall assume that in the present case also, the correct infrared limit is obtained by setting $S(A) = 0$. Moreover once one sets $S(A) = 0$, the present calculation reduces to the calculation of the infrared limit in Landau gauge, where one has $S_{\rm YM}(A) = 0$, and we may use directly the solution of [9], where d now represents the dimension of space instead of the dimension of space-time.

We briefly outline how the solution was obtained in [9]. The crucial point is that a solution was sought for which the ghost propagator, $\tilde{G}(k)$, is more singular than $1/k^2$ at $k =$

0. This property has been called the “horizon condition”, and it triggers the confinement scenario in Coulomb gauge. The horizon condition holds because the Gribov region Ω is bounded in every direction and, in a space of high dimension such as configuration space, entropy favors population density near the Gribov horizon $\partial\Omega$. At the horizon $M^{-1}(A^{\rm tr})$ is singular, and this enhances the ghost propagator $G(x-y) = \langle g_0 (M^{-1})_{xy} \rangle$ at large separation or small k. The coupled DS equations for the gluon and ghost propators were solved in [13], and [9], taking the tree-level expression for the ghost-gluon vertex, and imposing the transversality condition $\partial_i A_i = 0$ on-shell.

5. Calculation of color-Coulomb potential

To calculate $V_{\rm coul}(R)$ we write (1.2) as

$$V_{\rm coul}(x-y)\delta^{ab} = \int d^3z \ \langle \ \mathcal{G}^{ac}(x,z;A^{\rm tr}) \ (-\partial_z^2) \ \mathcal{G}^{cb}(z,y;A^{\rm tr}) \ \rangle, \tag{5.1}$$

and $\mathcal{G}^{ab}(x,y;A^{\rm tr}) \equiv g_0[M^{-1}(A^{\rm tr})]^{ab}_{xy}$. Its expectation-value, $G(x-y)\delta^{ab} = \langle \mathcal{G}^{ab}(x,y;A^{\rm tr}) \rangle$, is the ghost propagator. We separate the expectation-value of the product in (5.1) into disconnected and connected parts,

$$V_{\rm coul}(x-y) = \int d^3z \ G(x-z) \ (-\partial_z^2) \ G(z-y) + V_{\rm con}(x-y), \tag{5.2}$$

which reads, upon fourier transformation,

$$\tilde{V}_{\rm coul}(k) = k^2 \tilde{G}^2(k) + \tilde{V}_{\rm con}(k). \tag{5.3}$$

The infrared asymptotic form of the gluon and ghost propagators depends on two infrared critical exponents,

$$\begin{aligned} D^{\rm as}(k^2) &= \frac{b_D}{(k^2)^{1+\alpha_D}} \\ G^{\rm as}(k^2) &= \frac{b_G}{(k^2)^{1+\alpha_G}} \ . \end{aligned} \tag{5.4}$$

By equating like powers of momentum in either the gluon or the ghost DS equation, one obtains in either case the same relation $\alpha_D + 2\alpha_G = -(4-d)/2$. We use this equality to eliminate α_D, in favor of $\alpha \equiv \alpha_G$. In the infrared limit, only the ghost loop contributes to the DS equation for the gluon which reads gives $(b_D b_G^2)^{-1} = I_D(\alpha, d)$, where by eq. (A6) of [9]

$$I_D(\alpha, d) = \frac{N}{2 \ (4\pi)^{d/2}} \ \frac{\Gamma(2\alpha + 1 - d/2) \ \Gamma^2(-\alpha + d/2)}{\Gamma^2(1+\alpha)\Gamma(d-2\alpha)}. \tag{5.5}$$

The only process that contributes to the DS equation for the ghost propagator is emission and absorption of a gluon. In the infrared limit this gives $(b_D b_G^2)^{-1} = I_G(\alpha, d)$, where by eq. (A17) of [9]

$$I_G(\alpha, d) = \frac{N\ (d-1)}{2\ (4\pi)^{d/2}} \frac{\pi}{\sin(\pi\alpha)} \frac{\Gamma(2\alpha+1)\ \Gamma(-\alpha+d/2)}{\Gamma^2(\alpha+1)\ \Gamma(-2\alpha+d/2)\ \Gamma(\alpha+1+d/2)}. \tag{5.6}$$

We must solve

$$I_D(\alpha, d) = I_G(\alpha, d) \tag{5.7}$$

to find the infrared critical exponent $\alpha = \alpha(d)$.

6. Discussion of solution

We are interested in spatial dimension $1 < d \leq 3$. The integral for $I_D(\alpha, d)$ converges for α in the interval $\frac{1}{4}(d-2) < \alpha < \frac{1}{2}d$, and $I_D(\alpha, d)$ is positive in this interval and diverges at the end-points. The integral for $I_G(\alpha, d)$ converges for α in the interval $0 < \alpha < 1$, and it diverges at the end-points. However $I_G(\alpha, d)$ changes sign in this interval at $\alpha = \frac{1}{4}d$, and is positive only for $0 < \alpha < \frac{1}{4}d$. Thus we look for solutions for α in the range $\max[0, \frac{1}{4}(d-2)] \leq \alpha \leq \frac{1}{4}d$.

First take d in the interval $1 < d < 2$. We have $\frac{1}{4}(d-2) < 0$ so we restrict our consideration to the interval $0 < \alpha < \frac{1}{4}d$. From the values at the end-points it follows that there are an odd number of solutions, and from numerical plots one sees that there is precisely one solution $\alpha(d)$ for $1 < d < 2$. At $d = 1$, $I_G(\alpha, d)$ vanishes because of the coefficient $(d-1)$ in (5.6). This coefficient occurs because the ghost emits and aborbs a gluon whose propagator contains a d-dimensional transverse projector, which vanishes at $d = 1$. (The coefficient $d-1$ is an exact property of the DS equation of the ghost, and holds also in a more refined evaluation.) In contrast, $I_D(\alpha, d)$ is finite at $d = 1$ because a coefficient $d-1$ has been factored out of both sides of the DS equation for the gluon propagator. So $\alpha(d)$ vanishes linearly with $(d-1)$, and from (5.7) one obtains in the limit $d \to 1$,

$$\alpha(d) \to \frac{2}{\pi^2}(d-1) \approx 0.20264\ (d-1). \tag{6.1}$$

In fact this formula fits a solution $\alpha(d)$ of (5.7) to about 2 % accuracy in the entire interval $1 \leq d \leq 4$. At $d = 2$ the exact solution is given by

$$\alpha(2) = \frac{1}{5}, \tag{6.2}$$

which differs from (6.1) by 1.3 %. The fitting formula,

$$\alpha_{f1}(d) = \frac{1}{5}(d-1), \tag{6.3}$$

represents a solution $\alpha_1(d)$ to (5.7) with about 1 % accuracy in the interval $1 \leq d \leq 4$.

Now consider spatial dimension $2 \leq d \leq 3$. We have $0 \leq \frac{1}{4}(d-2)$, so we seek a solution in the interval $\frac{1}{4}(d-2) < \alpha < \frac{1}{4}d$. From the values at the end-points, there are now an even number of solutions, and from numerical plots it appears that there are precisely two distinct real roots $\alpha_1(d)$ and $\alpha_2(d)$ for $2 < d \leq 3$, except possibly at $d = d_c \approx 2.662$ where there appears to be one root at $\alpha(2.662) \approx 0.33095$, which is thus a crossing point of the two roots. At the crossing point (6.3) gives $\alpha_{f1}(2.662) = 0.33240$, which is accurate to 0.5 %. At $d = 2$ the two roots are given by $\alpha_1(2) = \frac{1}{5}$ and $\alpha_2(2) = 0$. Only the root $\alpha_1(2) = \frac{1}{5}$ matches the one root in the interval $1 < d < 2$, so it is the physical one, and the root $\alpha_1(2) = 0$ is spurious. Thus for $2 \leq d < d_c \approx 2.662$, the larger root is the physical one. For values $d > d_c$ above the crossing point we do not know which of the two roots is physical. At $d = 3$ the equality (5.7) simplifies to $\frac{32\alpha(1-\alpha)[1-\cot^2(\pi\alpha)]}{(3+2\alpha)(1+2\alpha)} = 1$, with roots

$$\alpha_1(3) \approx 0.3976; \qquad \alpha_2(3) = \frac{1}{2}. \tag{6.4}$$

The fitting formula (6.3) gives $\alpha_{f1}(3) = 0.4$, which agrees with the first root to about 1 %. At $d = 4$, there are two roots, $\alpha_1(4) = \frac{93-\sqrt{1201}}{98} \approx 0.5953$, and $\alpha_2(4) = 1$. The fitting formula (6.3) gives $\alpha_{f1}(4) = 0.6$, still accurate to 1 %. The fitting formula for the second root

$$\alpha_{f2}(d) = \frac{1}{2}(d-2) \tag{6.5}$$

is exact at $d = 2$, 3, and 4. The two fitting formulas cross at $d = \frac{8}{3} \approx 2.666$ and $\alpha = \frac{1}{3} \approx 0.333$.

From the relation $\alpha_D + 2\alpha_G = \frac{1}{2}(d-4)$, and the fitting formula (6.3), we obtain the critical exponent of the gluon propagator, $\alpha_D(d) \approx -\frac{3}{2} + \frac{1}{10}(d-1)$. This has a rather weak dependence on the spatial dimension d and gives a gluon propagator $D(k)$ that vanishes at $k = 0$ for $1 \leq d \leq 3$. Thus the would-be transverse physical gluon does not appear in the spectrum.

The critical exponent of the color-Coulomb potential is defined by

$$\tilde{V}^{\rm as}_{\rm coul}(k) = \frac{1}{(k^2)^{1+\alpha_V}}, \tag{6.6}$$

Suppose for simplicity that we neglect the connected term in (5.3) in the infrared asymptotic limit, leaving for another occasion an evaluation of this term. Then we have in this limit

$$\tilde{V}^{\rm as}_{\rm coul}(k) = \frac{b_G^2}{(k^2)^{1+2\alpha_G}}, \tag{6.7}$$

and we obtain for the infrared critical exponant of the color-Coulomb potential $\alpha_V = 2\alpha_G$. The color-Coulomb potential is given at large R by $V_{\rm coul}(R) \sim R^{2-d+2\alpha_V}$. If one uses the simple fitting formula (6.3) for α_G, one gets $\alpha_V = \frac{2}{5}(d-1)$, and $V_{\rm coul}(R) \sim R^{[1-0.2(d-1)]}$, which is a linear potential plus a rather weak dependence on d for $1 \leq d \leq 3$. For comparison we note that if instead $\alpha_G(d) = \frac{1}{4}\ (d-1)$, which is not so different from our solution, then one gets for the critical exponent of the gluon $\alpha_D = -\frac{3}{2}$, and of the color-Coulomb potential $\alpha_V = \frac{1}{2}(d-1)$. This gives an exactly linear potential $V_{\rm coul}(R) \sim R$, asymptotically at large R.

The second solution at $d = 3$, namely $\alpha_2(3) = \frac{1}{2}$, yields $\tilde{V}^{\rm as}_{\rm coul}(k) \sim 1/k^4$, which gives an exactly a linearly rising color-Coulomb potential. However this success must be regarded as partly accidental because our solution does not give an exactly linear potential at $d = 2$, and a correct calculation should work for both $d = 2$ and $d = 3$.

The deviation from a linear potential should not be regarded as a failure of the approach because our truncation scheme requires making an educated guess for the ghost-gluon vertex. We have chosen the tree-level vertex, but different choices give slightly different critical exponents [10]. So in our approach there is an inherent uncertainty in the critical exponent of the color-Coulomb potential. Moreover a correction to (6.7) may result from a more accurate evaluation of (5.3). However, granted these limitations, our results are at least qualitatively correct, and capture the essential features of a confining theory. We used the horizon condition which is the qualitative requirement that the ghost propagator be enhanced in the infrared compared to $1/k^2$. The dynamics of the DS equation then determine the infrared critical exponents of the ghost and gluon propagators and of the color-Coulomb potential. These are consistent with the confinement scenario in Coulomb gauge, which requires an infrared suppressed gluon propagator and a long-range color-Coulomb potential. They are also in at least qualitative agreement with numerical studies [14], [5]. The color-Coulomb potential we have obtained is confining and not far from linear for $1 \leq d \leq 3$.

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